Field Theoretical Characterization of Microwave Cavities

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ABSTRACT

A cylindrical method of lines (CMoL) has been developed to calculate waveguide cavities with homogeneous and inhomogeneous media. The advantage of the CMoL is that only two out of three space variables need to be discretized, while for the third direction the Helmholtz equation can be solved analytically. The CMoL is especially suitable for the analysis and design of cylindrical/rectangular cavities filled with dielectric blocks of arbitrary shape. Results are presented for a variety of resonator structures. Of particular interest is the resonant frequency calculation of a dielectric rod of varying diameter within a rectangular cavity.

INTRODUCTION

Microwave filters and multiplexers consist of several cavities which may be of complex shape. Characterization and design of these cavities is not a trivial task [1-3]. Commercially available software either uses approximations with sometimes insufficient accuracy or requires computational resources which are only available on supercomputers. Resonator structures that are of interest here are of rectangular or cylindrical shape containing dielectric blocks which are of cylindrical or rectangular cross-section, respectively. From the numerical point of view, cavities containing substructures whose contours are described by a different coordinate system than the cavity contour itself are difficult to handle.

To avoid most of the drawbacks of hitherto known numerical techniques like the finite element method (3D-discretization, spurious solutions), finite difference method (3D-discretization) or mode matching method (relative convergence, coupling integrals difficult to solve when mixed coordinates are involved), we have developed the 3D method of lines in cylindrical coordinates (CMoL). In an earlier paper we have shown that the CMoL is very flexible and computationally very efficient in the analysis of 2D eigenvalue problems [4]. For resonator calculations the CMoL must be extended to include an additional space variable. The advantage of the CMoL over other space discretization methods is that for the 3D problem only a 2D discretization is necessary. For the third space variable an analytical solution to the Helmholtz equation can be found. This saves not only computer memory space but also makes the algorithm computationally very efficient. To demonstrate the efficiency and accuracy of the 3D CMoL, results will be shown for rectangular and circular cavities partially filled with cylindrical dielectric blocks.

THEORY

To describe rectangular and cylindrical cavities with the same algorithm it is useful to express the fields components Ez and Hz in cylindrical coordinates by using the scalar potentials $\phi_{e,h}\exp\{j\omega t\}$ which satisfy the Helmholtz equation in the polar coordinate system

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)+\frac{\partial^2\phi}{r^2\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}+k_0^2\phi = 0 \tag{1}$$

All other field components can be derived from

$$E = \frac{1}{j\omega\varepsilon}\nabla\times\nabla(\phi_e\hat{z}) - \nabla(\phi_h\hat{z}) , \tag{2}$$

$$H = \frac{1}{j\omega\mu}\nabla\times\nabla(\phi_h\hat{z}) + \nabla(\phi_e\hat{z}) , \tag{3}$$

The generalized structure in Fig. 1 is then uniformly discretized by radial lines in the transverse plane, expressed by

$$\phi_{ki} = \phi_{1i} + (k-1)\,h_\theta \quad (k=1,2,\ldots,N_\theta),$$

and in the longitudinal direction by

$$\phi_{ki} = \phi_{k1} + (i-1)\,h_z \quad (i=1,2,\ldots,N_z),$$

with h_θ being the angular spacing between the lines in θ-direction and h_z the spacing between lines in z-direction. After the discretization we find the first order finite difference operator for the θ-direction as

$$h_\theta \frac{\partial \bar{\phi}_{e,h}}{\partial \theta} \cong [D]^{\theta}_{e,h} \bar{\phi}_{e,h},$$

and for the z direction

$$h_z \frac{\partial \bar{\phi}_{e,h}}{\partial z} \cong [D]^{z}_{e,h} \bar{\phi}_{e,h},$$

where $[D]^{\theta,z}$ represents bi-diagonal matrices. The second order finite difference operator for the θ-direction is found to be

$$h^2 \frac{\partial^2 \bar{\phi}_e}{\partial \theta^2}\bigg|_k = (-[D]^t)\,[D]\,\bar{\phi}_e = [P]\,\bar{\phi}_e,$$

$$h^2 \frac{\partial^2 \bar{\phi}_h}{\partial \theta^2}\bigg|_k = [D]\,(-[D]^t)\,\bar{\phi}_h = [P]\,\bar{\phi}_h,$$

and for the z-direction

$$h_z^2 \frac{\partial^2 \bar{\phi}_e}{\partial z^2}\bigg|_i = \bar{\phi}_e\left(-[D]^t_e\right)[D]_e = \bar{\phi}_e\,[Q]^t_e,$$

$$h_z^2 \frac{\partial^2 \bar{\phi}_h}{\partial z^2}\bigg|_i = \bar{\phi}_h\,[D]^t_h\,(-[D]_h) = \bar{\phi}_h\,[Q]^t_h,$$

where $[P] = [D](-[D]^t)$ can be factorized by an orthogonal matrix $[T_\theta]$ as

$$[P] = [T_\theta][\lambda][T_\theta]^t.$$

Similarly, an orthogonal transformation matrix $[T_z]$ can be found to diagonalize [Q] as

$$[T_z]^t[Q]^t[T_z] = \mathrm{diag}\{\delta_k\},$$

with $\mathrm{diag}\{\beta_k\}$ being the eigenvalues of $[T_z]$. The discretized Helmholtz equation is then given as

$$\frac{d}{r\,dr}\left(r\frac{d\bar{\phi}}{dr}\right) + k_0^2\bar{\phi} - \frac{[P]\,\bar{\phi}}{r^2 h_\theta^2} - \frac{\bar{\phi}\,[Q]^t}{h_z^2} = 0 \tag{4}$$

which represents an ordinary coupled differential equation. To decouple this equation the matrices [P] and [Q] must be diagonalized. This is possible by multiplying (4) from left and right with $[T_z]$ and $[T_\theta]$. The transformed potential then reads as

$$[\varphi] = \left[T_z^{e,h}\right]^t \bar{\phi} \left[T_\theta^{e,h}\right] \tag{5}$$

This transforms the discretized and coupled Helmholtz equation (4) into a set of decoupled and ordinary Bessel differential equations

$$\frac{d}{r\,dr}\left(r\frac{d\varphi_{ki}}{dr}\right) + \left(\chi_{ik}^2 - \frac{\mu_k^2}{r^2}\right)\varphi_{ki} = 0 \tag{6}$$

$$\chi_{ik}^2 = \left(k_0^2 + \frac{\delta_{ii}}{h_z^2}\right),\quad \mu_k = \frac{2\sin(\alpha_k/2)}{h} \tag{7}$$

In every uniform region, a solution of equ. (6) may be written as a superposition of Bessel functions of μ_k-order

$$\varphi_{ki} = A_k J_{\mu_k}(\chi_{ik} r) + B_k N_{\mu_k}(\chi_{ik} r) \tag{8}$$

It should be noticed that when a particular subregion contains the origin r=0, B_k must be zero since $N_{\mu k}$ is singular. After equ. (6) is solved in every uniform region (non-uniform region can also be solved by using a Sturm-Liouville equation), the potentials $\bar{\phi}$ can be obtained by an inverse transformation in equ. (5) for $\bar{\phi}_e$ and $\bar{\phi}_h$ respectively.

Inhomogeneous Structures

After the Helmholtz equations are solved in each uniform region, the fields at the interfaces between the regions are matched in order to solve the whole structure. The tangential fields at interfaces between neighboring regions are obtained from equs. (2) and (3) for ϕ_e and ϕ_h.

From the discretized potentials and (2) and (3) we obtain the coupled field continuity equation at the interface between subregions. Applying also here the orthogonal matrices $[T_z]^t$ and $[T_\theta]$, the equations are diagonalized to give a set of uncoupled continuity equations. Finally, transforming these potentials from the boundary of a subregion into the interface a relationship between the tangential fields E_z, E_θ and the surface currents densities J_z, J_θ in the interface plane can be obtained. Transforming potential functions and the discretized tangential fields back into the original domain yields

$$[Z]\begin{bmatrix} J_z \\ J_\theta \end{bmatrix} = \begin{bmatrix} E_z \\ E_\theta \end{bmatrix}.$$

Applying the condition of zero current in the dielectric interfaces, the final eigenvalue equation reads as

$$[Z]\begin{bmatrix} J_z \\ J_\theta \end{bmatrix} = 0,$$

which must be solved for the zeros of the determinant of matrix [Z]

$$det\{[Z]\} = 0 \quad (9)$$

All resonant frequencies of the cavities can be obtained from (9). From these solutions also the field components and current densities can be calculated.

Homogeneous Cavities

For homogeneous structures, the eigenvalue problem can be greatly simplified since there is no dielectric interface, the continuity condition required for inhomogeneous structures need not to be considered. Using equs. (2), (3) and (8), we can obtain the tangential fields ϕ_e and ϕ_h (proportional to E_z and H_z) on the cavity surface contour. Introducing the boundary conditions of $\bar{\phi} = 0$ (ϕ_e for TM modes and ϕ_h for TE modes) on the shielding contour leads directly to the eigenvalue equation as

$$\det\{[G]\} = 0$$

NUMERICAL RESULTS

The 3D CMoL has been tested for circular (Fig. 2) and rectangular (Fig. 3) cavities and compared to analytical solutions. It was found that the convergence of the method depends on the specific type of mode considered. For example, the analytical solution for the TM_{010} mode in Fig. 2 is already reached with only 20-25 lines, while the TE_{111} mode requires approximately 35 lines. For the rectangular resonator (Fig. 3) a larger number of lines is needed because of the approximation of the rectangular boundary by radial discretization lines. Loading the cylindrical cavity with a cylindrical dielectric block of varying radius reduces the resonant frequency of the first resonator mode (Fig. 4). The same effect is obtained when we insert a cylindrical dielectric block into a rectangular cavity (Fig. 5). The interesting fact here is that the structure contains subregions which are described by a rectangular (rectangular cavity) and a cylindrical coordinate system (dielectric rod). To discretize this 3D structure requires approximately 50 lines in the a-b-plane and 12-lines in the longitudinal direction (L). The computation time per frequency on an IBM RS 6000 (530) is about 1 minute. However, since the computer code is not optimized yet, we expect a ten times improvement once this is done. Fig. 6 illustrates the fundamental mode resonant frequency for a non-uniform dielectric rod with varying height of the partially thicker diameter. It should be noted that the transition between the thin rod diameter (r_0) and the thick section is made smoothly rather than abruptly. This is done to demonstrate the flexibility of the 3D CMoL.

CONCLUSION

A 3D cylindrical method of lines has been presented for the analysis and design of microwave resonators of arbitrary shape. Cylindrical and rectangular cavities are calculated with and without partial dielectric filling. The computational resources required are found to be much less than for other numerical approaches. This was demonstrated by analyzing a rectangular resonator partially filled with a non-uniform cylindrical dielectric rod with smoothly changing diameter.

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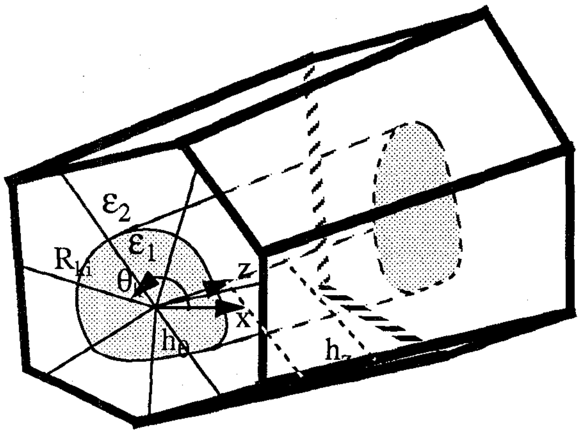

Fig. 1 3D MoL discretization in a cylindrical coordinate system.

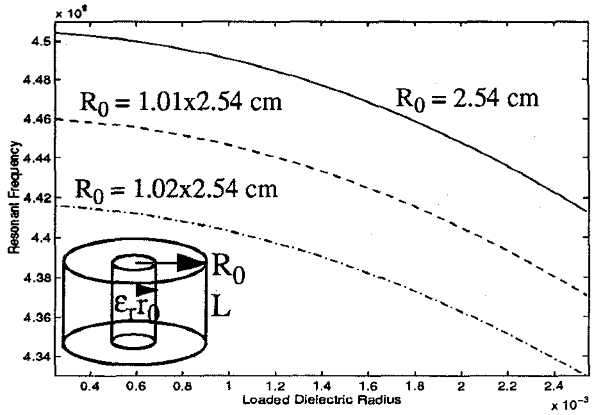

Fig. 4 Resonant frequencies of the TM_{010} mode for a dielectric-loaded circular waveguide cavity, $L = R_0 = 2.54$ cm, 1.01x2.54 cm, 1.02x2.54 cm, $r_0 = 0.254$ mm ~ 2.54 mm.

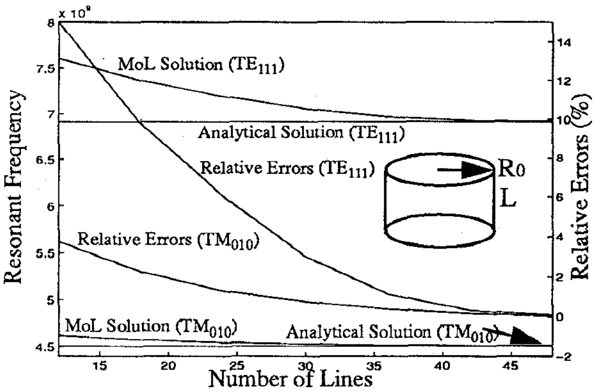

Fig. 2 CMoL calculation of resonant frequencies of TE_{111} and TM_{010} modes compared to the analytical solutions for a circular waveguide, $L = R_0 = 2.54$ cm.

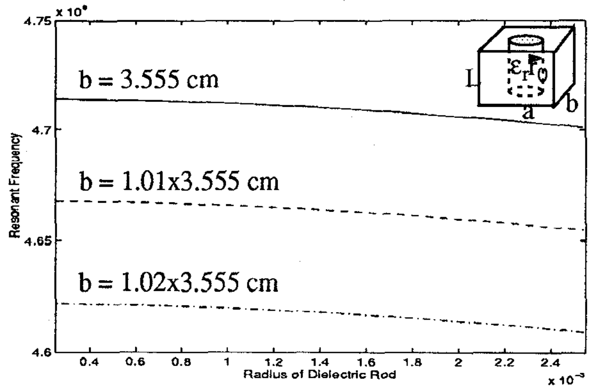

Fig. 5 Resonant frequencies of the TM_{110} mode for a dielectric-loaded rectangular waveguide cavity, $L = b = a/2 = 3.555$ cm, 1.01x3.555 cm, 1.02x3.555 cm, $r_0 = 0.254$ mm ~ 2.54 mm.

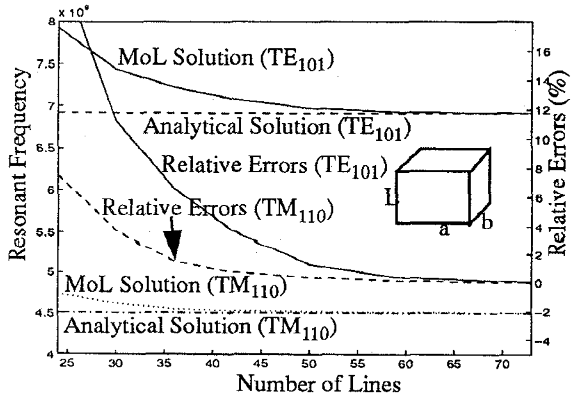

Fig. 3 CMoL calculation of resonant frequencies of TE_{101} and TM_{010} modes compared to the analytical solutions for a rectangular waveguide, $L = b = a/2 = 3.555$ mm.

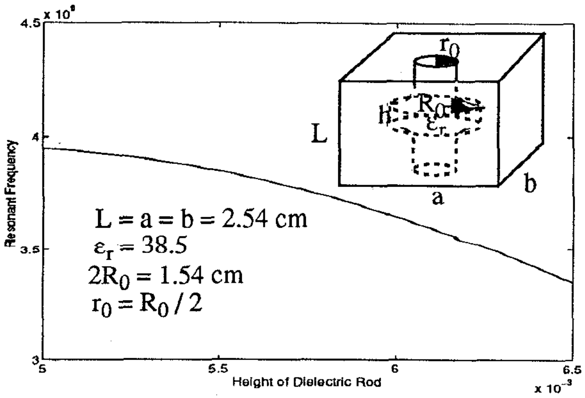

Fig. 6 Resonant frequencies of the fundamental mode in a dielectric loaded waveguide cavity, a=b=L=2.54 cm, $R_0 = 1.54$ cm, $r_0 = R_0/2$, $\varepsilon_r = 38.5$.